F. anal. 17.3.2025

- ~~Slides (?) (oversikt of F-transf.?)~~, info.

Application: Central limit thm. (CLT)

A. ~~Some probability theory~~

Prob. density f'n (pdf): on $\mathbb{R}$: $f \in C_m(\mathbb{R})$ (or $L^1(\mathbb{R})$) s.t.

$$f \geq 0 \quad \text{and} \quad \int_{\mathbb{R}} f = 1$$

A rnd. var. (r.v.) $\underset{\text{a set}}{X : \Omega \to \mathbb{R}}$ has pdf f, $X \sim f$, if

$$P(X \in [a,b)) = \int_a^b f(x)\,dx \quad \forall\ -\infty < a < b < \infty$$

probability

Exp. Var $N(\mu, \sigma^2)$

R.v.'s $X_1, \dots, X_n$ w. pdf's $f_1, \dots, f_n$ are independent if

$$P(X_1 \in [a_1, b_1), \dots, X_n \in [a_n, b_n)) = \left(\int_{a_1}^{b_1} f_1\right) \cdots \left(\int_{a_n}^{b_n} f_n\right),$$

and they are indep., identically distributed (i.i.d.) if also $f_1 = \dots = f_n$

Obs 1: $\{X_k\}_k$ i.i.d. then $E(X_1) = E(X_2) = \cdots$, $Var(X_1) = Var(X_2) = \cdots$

R.v.'s $\{X_k\}_{k=1}^{\infty}$ i.i.d. if $\{X_k\}_{k=1}^{n}$ i.i.d for any $n \in \mathbb{N}$.

Lem 1: $f_1, f_2 \in C_m(\mathbb{R})$ (or $L^1(\mathbb{R})$) pdf's. X_1, X_2 indep. r.v.'s

$$X_1 \sim f_1, \ X_2 \sim f_2 \implies X_1 + X_2 \text{ is r.v., } \sim f_1 * f_2 \ (\dots)$$

Pf.:

$$P(X_1+X_2 \in [a,b)) = \iint_{a \le x_1+x_2 \le b} \underbrace{f_{(X_1,X_2)}(x_1,x_2)}_{\underset{\text{indep.}}{=} f_1(x_1) f_2(x_2)} \, dx_1 \, dx_2$$

$x_1 + x_2 = y$

$$= \int_a^b \int_{\mathbb{R}} f_1(y-x_2) f_2(x_2) \, dx_2 \, dy = \int_a^b f * g \qquad \square$$

$dy = dx_1$

Cor. 1: $f \in C_m(\mathbb{R})$ $(L^1(\mathbb{R}))$ pdf, $\{X_k\}_{k=1}^n$ i.i.d. r.v.'s, $X_1 \sim f$, then

$$S_n = \sum_{k=1}^n X_k \sim f^{*n} = \underbrace{f * \cdots * f}_{n \text{ times}}$$

Expectation: $E(X) = \int_{\mathbb{R}} x f(x) \, dx$ $(X \sim f)$

Variance: $Var(X) = E[(X - E(X))^2] = \int_{\mathbb{R}} (x - E(X))^2 f(x) \, dx$

X Gaussian/normal r.v., $X \sim N(\mu, \sigma^2)$, if

$$X \sim \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (= K_{2\pi\sigma^2}(x-\mu) \in S(\mathbb{R}))$$

$X \sim N(\mu, \sigma^2) \Rightarrow E(X) = \mu, \; Var(X) = \sigma^2$

B. CLT

Thm. 1: CLT - version 1

If f pdf, $\int_{\mathbb{R}} x f(x) \, dx = 0$, $\int_{\mathbb{R}} x^2 f(x) \, dx = 1$, then

$$\lim_{n \to \infty} \int_a^b \sqrt{n} f^{*n}(\sqrt{n} x) \, dx = \int_a^b \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{K_{2\pi}(x)} \, dx \quad \forall -\infty < a < b < \infty$$

Thm. 2: CLT – version 2

If $\{X_k\}_{k=1}^{\infty}$ i.i.d. r.v.'s, X_1 has pdf, $E(X_1)=0$, $Var(X_1)=E(X_1^2)=1$, then

$$\lim_{n\to\infty} P\left(\tfrac{1}{\sqrt{n}}S_n = \tfrac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k \in [a,b)\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

Rem. 1:

a) $Var\left(\tfrac{1}{\sqrt{n}}S_n\right) = E\,\tfrac{1}{n}\left(\sum X_k\right)^2 \overset{\text{indep.}}{=} \tfrac{1}{n}\sum E(X_k^2) = 1$, $E\left(\tfrac{1}{\sqrt{n}}S_n\right)=0$

a) $\tfrac{1}{\sqrt{n}}S_n \xrightarrow[n\to\infty]{\text{distribution}} X \sim N(0,1)$ (approx $N(0,1)$ for $n>>1$)

b) $X\sim f \overset{\text{Cor.1}}{\Longrightarrow} S_n \sim f^{*n}$

$$\Rightarrow P\left(\tfrac{1}{\sqrt{n}}S_n\in[a,b)\right) = P\left(S_n\in[\sqrt{n}a,\sqrt{n}b)\right)$$

$$= \int_{\sqrt{n}a}^{\sqrt{n}b} f^{*n}(x)\,dx \overset{x=\sqrt{n}y}{=} \int_a^b \sqrt{n}\,f^{*n}(\sqrt{n}y)\,dy$$

Hence Thm 1 $\Leftrightarrow$ Thm 2

Pf. Thm. 1:

Lem. 2: If f as in Thm. 1 and $\varphi\in S(\mathbb{R})$, then

$$(1)\quad \lim_{n\to\infty}\underbrace{\int_{\mathbb{R}}\sqrt{n}\,f^{*n}(\sqrt{n}x)\,\varphi(x)\,dx}_{=I_n} = \int K_{2\pi}(x)\,\varphi(x)\,dx$$

k(Pf. for $f, x^2 f \in C_m$:

1) $$I_n = \int_{\mathbb{R}}\sqrt{n}\,f^{*n}(\sqrt{n}x)\Big[\underbrace{\int_{\mathbb{R}}\check{\varphi}(\xi)\,e^{-2\pi i\xi x}\,d\xi}_{=\varphi(x)\ (\mathcal{F}.\text{-inv.},\ \varphi\in S(\mathbb{R}))}\Big]\,dx$$

$$\underset{\text{of int.}}{\overset{\text{chng. order}}{=}} \int_{\mathbb{R}}\check{\varphi}(\xi)\Big[\underbrace{\int_{\mathbb{R}}\sqrt{n}\,f^{*n}(\sqrt{n}x)\,e^{-2\pi i\xi x}\,dx}\Big]\,d\xi$$

$f^{*n}\in C_m$, $\check{\varphi}\in S(\mathbb{R})$

$$= \mathcal{F}[\sqrt{n}\,f^{*n}(\sqrt{n}x)](\xi) \overset{\text{scaling}}{=} \mathcal{F}[f^{*n}]\left(\tfrac{\xi}{\sqrt{n}}\right) \overset{\hat{*}}{=} \left(\hat{f}\left(\tfrac{\xi}{\sqrt{n}}\right)\right)^n$$

D9: 02/03

2) $\hat{f}\left(\frac{\xi}{\sqrt{n}}\right) = 1 - \frac{2\pi^2\xi^2}{n}(1+o(1))$ as $n \to \infty$,

w.r.t. loc. unif. conv. ;

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(2) $$\hat{f}\left(\frac{\xi}{\sqrt{n}}\right) = \int_{\mathbb{R}} \underbrace{e^{-2\pi i x \frac{\xi}{\sqrt{n}}}}_{= e^{-iy},\ y = 2\pi x \frac{\xi}{\sqrt{n}}} f(x)\, dx$$

$$e^{-iy} \overset{\text{Taylor's thm}}{=} 1 - iy + \frac{1}{2}(iy)^2 \underbrace{(e^{-i\eta})}_{=(e^{-is})''_{s=\eta}},\ \eta \in (0,y)$$

(3) $$= 1 - iy - \frac{1}{2}y^2\Big(1 + \underbrace{(e^{-i\eta} - 1)}_{= \tilde{E}(\eta)}\Big)$$

(4) $$|\tilde{E}(\eta)| \le 1 + 1,\quad |\tilde{E}(\eta)| \overset{\text{Taylor}}{\le} |\underset{=0}{\tilde{E}(0)} + \eta \cdot \underbrace{\tilde{E}'(\tilde{\eta})}_{= -e^{-i\tilde{\eta}}}| \le |\eta| \le |y| \le \frac{2\pi |x||\xi|}{\sqrt{n}}$$

Hence:

$$\hat{f}\left(\frac{\xi}{\sqrt{n}}\right) \overset{(2)}{\underset{(3)}{=}} \underset{=1}{\int f} - 2\pi i \xi \underbrace{\int x f}_{=0} - \frac{1}{2}\frac{4\pi^2\xi^2}{n}\underbrace{\int x^2 f}_{=1}$$

$$+ \frac{1}{2}\frac{4\pi^2\xi^2}{n} \underbrace{\int \underbrace{x^2 \underbrace{\tilde{E}(\eta)}_{\overset{(4)}{\le} \min\left(2, \frac{2\pi|x||\xi|}{\sqrt{n}}\right)} f(x)}_{=: F_n(x,\xi) \xrightarrow[n\to\infty]{} 0 \text{ loc. unif.}} dx}_{\xrightarrow[n\to\infty]{\text{"dom. conv."}} 0 \text{ loc. unif. in } \xi}$$

$[\, |F_n| \le 2x^2 f \in C_m \,]$

3) $J_n := \left[\hat{f}\left(\frac{\xi}{\sqrt{n}}\right)\right]^n \overset{2)}{=} \underbrace{\left[1 - \frac{2\pi^2\xi^2}{n}(1+\sigma(1))\right]^n}_{=J_n} \longrightarrow e^{-2\pi^2\xi^2}$ loc. unif. :

Let $s = 2\pi^2\xi^2$, $z = \frac{s}{n}(1+\sigma(1))$, then

$$J_n = e^{n\underbrace{\ln(1-z)}_{=-\sum_{n=1}^{\infty}\frac{z^n}{n}}} \underset{\text{thm.}}{\overset{\text{Taylor's}}{=}} e^{n(-z-\frac{1}{2}\eta^2)}, \quad \eta \in (0,z)$$

Since $|\eta^2| \leq |z|^2 = \frac{s^2}{n^2}O(1)$,

$$n(-z-\tfrac{1}{2}\eta^2) = -s - \sigma(1)s - \tfrac{1}{2}\tfrac{s^2}{n}O(1) \underset{n\to\infty}{\overset{2)}{\longrightarrow}} -s \text{ loc. unif.}$$

$$\Longrightarrow J_n = J_n(s) \longrightarrow e^{-s} \text{ loc. unif. } [s = 2\pi^2\xi^2]$$

4) Conclusion:

$$I_n \overset{1)}{\underset{3)}{=}} \int_{\mathbb{R}} \check{\varphi}(\xi)\, J_n(\xi)\, d\xi$$

(i) $\|J_n\|_\infty \leq \|\hat{f}\|_\infty^n \leq \|f\|_1^n = 1$

$\Longrightarrow |\check{\varphi}(\xi) J_n(\xi)| \leq |\check{\varphi}(\xi)| \in C_m \quad [\varphi \in S \Rightarrow \check{\varphi} \in S \subset C_m]$

(ii) $\check{\varphi}\, J_n \underset{n\to\infty}{\overset{3)}{\longrightarrow}} \check{\varphi}(\xi)\, e^{-2\pi^2\xi^2}$ loc. unif.

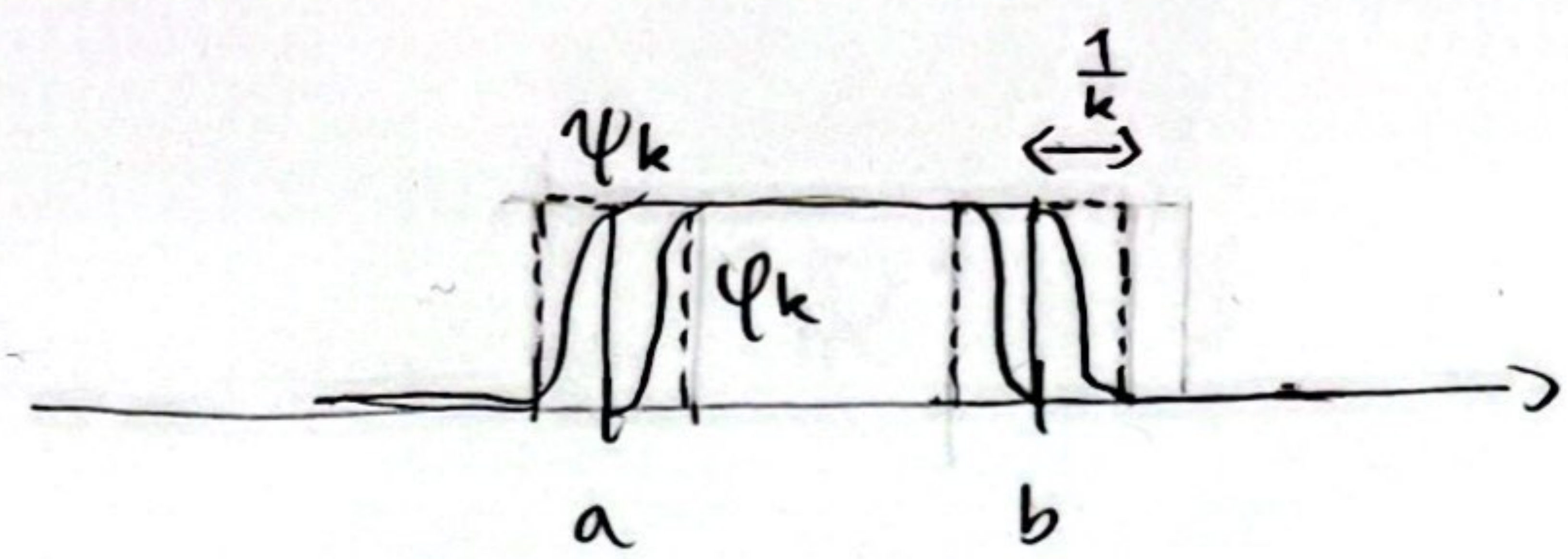
"Dominated conv" (prev.):

$$\Longrightarrow I_n = \int \check{\varphi}\, J_n \underset{n\to\infty}{\longrightarrow} \int \check{\varphi}\, \underbrace{e^{-2\pi^2\xi}}_{=\hat{K}_{2\pi}(\xi) = \check{K}_{2\pi}(-\xi)} = \int \check{\varphi}\, \check{K}_{2\pi} \underset{\substack{\uparrow \\ \text{Plancherel} \\ \varphi, K \in S(\mathbb{R})}}{=} \int \varphi\, K_{2\pi} \quad \square$$

Pf. of Thm. 1 for $f,\ x^2 f \in C_m$:

1) Take $\varphi_k, \psi_k \in S(\mathbb{R})$ s.t.

$$\chi_{[a+\frac{1}{k},\, b-\frac{1}{k}]} \le \varphi_k \le \chi_{[a,b]} \le \psi_k \le \chi_{[a-\frac{1}{k},\, b+\frac{1}{k}]}$$



~~2) By Lem. 2,~~

2) $$\int_a^b \sqrt{n}\, f^{*n}(\sqrt{n}x)\, dx \underset{\chi_{[a,b]}\le\psi_k}{\overset{f\ge 0}{\le}} \int_{\mathbb{R}} \sqrt{n}\, f^{*n}(\sqrt{n}x)\, \psi_k\, dx$$

$$\xrightarrow[n\to\infty]{\text{Lem. 2}} \int_{\mathbb{R}} K_{2\pi}\, \psi_k \underset{\psi_k \le \chi_{[a-\frac{1}{k},\, b+\frac{1}{k}]}}{\overset{K_{2\pi}\ge 0}{\le}} \int_{a-\frac{1}{k}}^{a+\frac{1}{k}} K_{2\pi}$$

Hence, since ok for $\forall k \in \mathbb{N}$,

$$\limsup_{n\to\infty} \int_a^b \sqrt{n}\, f^{*n}(\sqrt{n}x) \le \inf_{k\in\mathbb{N}} \int_{a-\frac{1}{k}}^{b+\frac{1}{k}} K_{2\pi} \overset{\text{chk.}}{=} \int_a^b K_{2\pi}$$

3) Similarly, using φ_k, (chk!)

$$\liminf_{n\to\infty} \int_a^b \sqrt{n}\, f^{*n}(\sqrt{n}x) \ge \sup_{k\in\mathbb{N}} \int_{a+\frac{1}{k}}^{b-\frac{1}{k}} K_{2\pi} = \int_a^b K_{2\pi}$$

4) By 2), 3), ~~lim ex., and $\lim_{n\to\infty} \int - = \int_a^b K_{2\pi}$ □~~

$$\limsup - \le \int_a^b K_{2\pi} \le \liminf -$$

$$\overset{\limsup \ge \liminf}{\Longrightarrow} \limsup(\ldots) = \liminf(\ldots) = \lim(\ldots) = \int_a^b K_{2\pi}$$ □

F. anal. 20.3.2025

- Slides (?), info (next week)

A. Preliminaries on $\mathbb{R}^d$

$\mathbb{R}^d$ vec. sp. of $x=(x_1,\dots,x_d)$, $x_i \in \mathbb{R}$.

Hilbert sp. w.

i.p. $(x,y) = x\cdot y = x_1 y_1 + \dots + x_d y_d$

norm $|x|^2 = (x,x) = x_1^2 + \dots + x_d^2$

Notation:

i) Multi-index $\alpha = (\alpha_1,\dots,\alpha_d)$, $\alpha_i \in \mathbb{N}\cup\{0\}$

$|\alpha| := \alpha_1 + \dots + \alpha_n$

ii) $x^\alpha = x_1^{\alpha_1}\dots x_d^{\alpha_d}$

iii) $\partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\dots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\dots\partial x^{\alpha_d}}$

Symmetries in $\mathbb{R}^d$:

(a) Translations: $x \mapsto x+h$

(b) Dilations: $x \mapsto \delta x$, $\delta > 0$

(c) Rotations: $R: \mathbb{R}^d \to \mathbb{R}^d$ c.f.

(i) $R(ax+by) = aR(x) + bR(y) \quad \forall x,y \in \mathbb{R}^d,\ a,b \in \mathbb{R}$

(ii) $|R(x)| = |x|, \quad \forall x \in \mathbb{R}^d$

Rem. 1: (a) (i) $\Leftrightarrow$ $R(x) = R \cdot x$ for matr. $R \in \mathbb{R}^{d \times d}$

(a) (ii) $\Leftrightarrow$ $R(x) \cdot R(y) = x \cdot y \ \forall x,y \in \mathbb{R}^d \overset{(i)}{\Leftrightarrow} R^T = R^{-1} \ (\in \mathbb{R}^{d \times d})$

$[(Rx, y) = (x, R^T y)]$

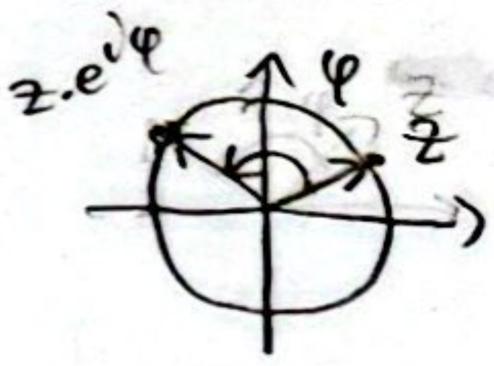
(b) $\det R = \pm 1$; R proper/improper rotation if $\det R = \pm 1$

$(\det I = \det RR^T = \det R \det R^T = (\det R)^2)$

Ex. 1:

(a) 2 rotations in $\mathbb{R}$: $x \mapsto x$, $x \mapsto -x$

(b) In $\mathbb{R}^2 \simeq \mathbb{C}$: $(z = x + iy)$



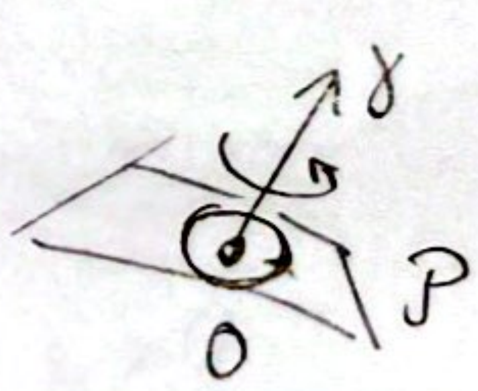
All proper rotations: $z \mapsto z e^{i\varphi}$ for $\varphi \in \mathbb{R}$

All improper — " —: $z \mapsto \bar{z} e^{i\varphi}$ for $\varphi \in \mathbb{R}$

[Exercise 11]

(c) In $\mathbb{R}^3$ (Euler):

R proper rotation $\Rightarrow \exists\ \gamma \in \mathbb{R}^3, |\gamma| = 1$ s.t.



(i) $R(\gamma) = \gamma$ ("axis of rotation")

(ii) the restriction $R|_P$ is an $\mathbb{R}^2$-rotation,

where the plane P is s.t. $0 \in P$ and $P \perp \gamma$

[Exercise 11]

R improper $\Leftrightarrow -R$ proper $[\det_{\mathbb{R}^3}(-R) = -\det_{\mathbb{R}^3}(R)]$

odd dim.

Ex. 2: $\{e_1, \dots, e_d\}$, $\{e_1', \dots, e_d'\}$ ONB in $\mathbb{R}^d$ and

orthonormal basis

(a) Def. $R: \mathbb{R}^d \to \mathbb{R}^d$ by $R(e_i) = e_i'$, $i = 1, \dots, d$ $[\Leftrightarrow R(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i e_i']$

lin. map

then ~~R lin.~~ $|R(x)| = |x|$, $\Rightarrow$ R rotation

(lin. transformation)
basis shift

(b) For any rotation R and ONB $(e_1, \dots, e_d)$,

$(e_1', \dots, e_d') := (Re_1, \dots, Re_d)$ is again an ONB

B. Integration in $\mathbb{R}^d$

Improper Riemann int.:

$$\underbrace{\int_{\mathbb{R}^d} f(x)\, dx}_{I} = \lim_{N \to \infty} \underbrace{\int_{|x| \le N} f(x)\, dx}_{I_N}$$

Exists if (prev.)

$$f \in C_m(\mathbb{R}^d) = \{ f \in C(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x|^{d+\varepsilon} |f(x)| < \infty \}$$

cont., moderate decrease

[then $\{I_N\}_{N \in \mathbb{N}}$ Cauchy in $\mathbb{R}$ (complete)]

Ex. 3: $e^{-|x|^2}$, $e^{-|x|}$, $(1+|x|^2)^{\frac{d+\varepsilon}{2}} \in C_m(\mathbb{R}^d)$ for $\varepsilon > 0$

Lem. 1: $f \in C_m(\mathbb{R}^d)$ (or $L^1(\mathbb{R}^d)$)

(a) $\int_{\mathbb{R}^d} f(x+h)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \forall h \in \mathbb{R}^d$

(b) $\delta^d \int_{\mathbb{R}^d} f(\delta x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \forall \delta > 0$

(c) $\int_{\mathbb{R}^d} f(R(x))\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \forall$ rotations R

Change of var.'s:

Thm. 1: $A, B \subset \mathbb{R}^d$ closed, bnd; $g: A \to B$ inv.; $g, g^{-1} \in C^1$; $f \in C(B)$

$$\Rightarrow \int_{g(A)} f(x)\,dx = \int_A f(g(y))\,\underbrace{|\det(Dg)(y)|}_{\text{Jacobian}}\,dy$$

["$x = g(y)$, $dx = |\det Dg(y)|\,dy$"] $\quad$ [$Dg = (\partial_i g_j)_{i,j}$ Jacobi matr.]

Pf. of Lem 1:

For $f \geq 0$:

(a) $$\int_{|x| \leq N-h} f(x)\,dx \leq \int_{|x| \leq N} f(x+h)\,dx \leq \int_{|x| \leq N+h} f(x)\,dx$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} f(x+h) \leq \int_{\mathbb{R}^d} f\,dx$$

General f: $f = f^+ - f^-$, $f^+ = \max(f, 0)$, $f^- = (-f)^+$.

(b) $$\delta^d \int_{|x| \leq N} f(\delta x)\,dx \overset{y = \delta x}{=} \int_{|y| \leq \delta N} f(y)\,dy, \text{ send } N \to \infty.$$

$dy_1 \cdots dy_d = \delta^d dx_1 \cdots dx_d$

(c) $$\int_{|y| \leq N} f(R(y))\,dy \overset{\text{Thm. 1}}{=} \int_{|y| \leq N} f(R(y))\,\underbrace{|\det(\underbrace{DR(y)}_{=R})|}_{=\pm 1}\,dy$$

$$\overset{\text{Thm. 1}}{\underset{x = R(y)}{=}} \int_{|R^{-1}x| \leq N} f(x)\,dx = \int_{|x| \leq N} f(x)\,dx, \text{ send } N \to \infty. \quad \square$$

$|Ry| = |y|$

$\Rightarrow |R^{-1}x| = |x|$

9:06

Spherical coordinates:

$\boxed{\text{Obs:}}$ $x \in \mathbb{R}^d \setminus \{0\} \Rightarrow x = |x| \cdot \frac{x}{|x|} = r \cdot \gamma$, $r = |x|$, $|\gamma| = 1$

(1) $$\int_{|x| \leq R} f(x)\,dx \overset{f \in C}{=} \int_{|\gamma| = 1} \int_0^R f(r\gamma)\,r^{d-1}\,dr\,d\sigma(\gamma)$$

where $d\sigma(\gamma)$ is surface element on unit sphere S^{d-1} in $\mathbb{R}^d$, defined from spherical coord's in terms of $(d-1)$ angles [see SS Appendix]

Obs: $f \in C_m(\mathbb{R}^d) \Rightarrow g(r) = f(\gamma r) r^{d-1} \in C_m(\mathbb{R})$:

$$\left[\sup_{r \in \mathbb{R}} |r|^{1+\varepsilon} |g(r)| = \sup_{r\in\mathbb{R}} |r|^{d+\varepsilon} |f(\gamma r)| \leq \sup_{x \in \mathbb{R}^d} |x|^{d+\varepsilon} |f(x)| < \infty \right]$$

Hence for $f \in C_m(\mathbb{R}^d)$, sending $R \to \infty$ in (1),

$$(2) \quad \int_{\mathbb{R}^d} f(x)\,dx = \int_{|\gamma|=1} \int_0^\infty f(r\gamma) r^{d-1}\,dr\,d\sigma(\gamma).$$

Ex. 3: $\mathbb{R}^2$

$(x,y) = (r\cos\theta, r\sin\theta)$

$$\iint_{|(x,y)|\leq R} f(x,y)\,dx\,dy \overset{(\text{Thm. 1})}{=} \int_0^{2\pi} \int_0^R f(r\underbrace{(\cos\theta,\sin\theta)}_{=\gamma\ (|\gamma|=1\,!)})\, r\,dr\,d\theta = I$$

Def. $d\sigma(\gamma)$ by:

$$\int_{|\gamma|=1} h(\gamma)\,d\sigma(\gamma) = \int_0^{2\pi} h(\cos\theta,\sin\theta)\,d\theta \quad \forall\, h \in C(\mathbb{S})$$

Then

$$I = \int_{|\gamma|=1} \int_0^R f(r\gamma)\, r\,dr\,d\sigma(\gamma)$$

Ex. 4: $\mathbb{R}^3$

$$(x_1,x_2,x_3) = r\overbrace{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}^{=:\gamma\ (|\gamma|=1\,!)}$$

$$\iiint_{|x|\leq R} f(x)\,dx \overset{(\text{Thm. 1})}{=} \int_0^{2\pi}\int_0^{\pi}\int_0^R f(r\gamma)\, r^2\,dr\,d\varphi\,d\theta = I$$

Def. $d\sigma(\gamma)$ by:

$$\iint_{|\gamma|=1} h(\gamma)\,d\sigma(\gamma) = \int_0^{2\pi}\int_0^{\pi} h(\gamma)\sin\theta\,d\varphi\,d\theta \quad \forall\, h \in C(\mathbb{S}^2)$$

Then

$$I = \int_{|\gamma|=1}\int_0^R f(r\gamma)\, r^{d-1}\,dr\,d\sigma(\gamma)$$

Exercise 11: Let $B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$, $\partial B_1 = \{x \in \mathbb{R}^d : |x| = 1\}$

Then $A_d = \text{area}(\partial B_1) = \int_{|y|=1} d\sigma(y) = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$

$V_d = \text{vol}(B_1) = \int_{|y|=1} \int_0^1 r^{d-1} dr\, d\sigma(y) = \dfrac{A_d}{d} = \dfrac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$

[Hint: Show $1 = \int_{\mathbb{R}^d} e^{-\pi |x|^2} dx = A_d \int_0^\infty e^{-\pi r^2} r^{d-1} dr$, $\Gamma = \cdots$]

C. The Fourier transform in $C_m(\mathbb{R}^d)$ (L^1)

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d$$

Well-def. for $f \in C_m(\mathbb{R}^d)$ (or $L^1(\mathbb{R}^d)$):

Prop. 1: $f \in C_m(\mathbb{R}^d)$ (or $L^1(\mathbb{R}^d)$)

$\Longrightarrow \hat{f} \in C_0(\mathbb{R}^d)$, $\|\hat{f}\|_\infty \leq \|f\|_1$, $|\hat{f}(\xi)| \xrightarrow[|\xi| \to \infty]{} 0$

Pf.: Similar to $\mathbb{R}^1$ case $\square$

Next
radial f'ns
Gaussians
Convolutions
good kernels
$\mathcal{F}$ in $S(\mathbb{R}^d)$
derivatives
$S \to S$
inversion
Plancherel
$\mathcal{F}$ in L^2
wavelets

Prop. 2: $f \in C_m(\mathbb{R}^d)$ (or $L^1(\mathbb{R}^d)$)

(a) $\mathcal{F}[f(x+h)](\xi) = e^{2\pi i \xi \cdot h} \hat{f}(\xi)$, $h \in \mathbb{R}^d$

(b) $\mathcal{F}[e^{-2\pi i x \cdot h} f(x)](\xi) = \hat{f}(\xi + h)$, $h \in \mathbb{R}^d$

(c) $\mathcal{F}[f(\delta x)](\xi) = \delta^{-d} \hat{f}(\delta^{-1}\xi)$, $\delta > 0$

(d) $\mathcal{F}[f(Rx)](\xi) = \hat{f}(R\xi)$, R rotation

Pf.: (a)–(c): Similar to $\mathbb{R}^1$ case (see Lem 1).

(d) $\int_{\mathbb{R}^d} f(Rx) e^{-2\pi i x \cdot \xi} dx \overset{\substack{y = Rx \text{ (Thm 1)} \\ dy = dx}}{=} \int_{\mathbb{R}^d} f(y) e^{-2\pi i R^{-1} y \cdot \xi} dy$, $R^{-1}y \cdot \xi = R^T y \cdot \xi = y \cdot R\xi$

Prop. 3: (a) If $\int_{\mathbb{R}^d} |f| < \infty$ and f cont. at x,

then $(K_\delta * f)(x) \longrightarrow f(x)$ as $\delta \to 0$.

(b) If $f \in C_m(\mathbb{R}^d)$, then $K_\delta * f \xrightarrow[\delta \to 0]{} f$ uniformly

[Pf as in $\mathbb{R}^1$].

go to p. 3

Radial f'ns:

f radial if $\exists f_0 : [0, \infty) \to \mathbb{C}$ s.t. $f(x) = f_0(|x|)$

Ex. 5: $e^{-\pi |x|^2}$, $\dfrac{1}{1+|x|^2}$

Lem. 1: f radial $\iff$ $f(Rx) = f(x)$ $\forall$ rotations R

Pf: $\Rightarrow$) $f(Rx) = f_0(|Rx|) = f_0(|x|) = f(x)$.

$\Leftarrow$) Def. $f_0(r) = f(x)$ if $|x| = r$.

well-def. since $|x_1| = |x_2| \Rightarrow \exists$ rotation R s.t. $x_1 = R x_2$, ~~R rotation~~

$\Rightarrow f(x_2) = f(R x_2) = f(x_1)$

Hence $f(x) = f_0(|x|)$ and f radial $\square$

Prop. 3: $f \in C_m(\mathbb{R}^d)$ (or L^1) radial

$\Rightarrow \hat{f}$ radial

Pf.:

$$\hat{f}(R\xi) \underset{\substack{\uparrow \\ \text{Prop 2d} \\ \text{last time}}}{=} \mathcal{F}[f(Rx)](\xi) \underset{\substack{\uparrow \\ f \text{ radial}}}{=} \mathcal{F}[f(x)](\xi) = \hat{f}(\xi)$$

$\Rightarrow \hat{f}$ radial by Lem. 1 $\square$